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Bertoluzza *et al.*'s metric as a basis for analyzing fuzzy data*

María Rosa Casals · Norberto Corral · María Ángeles Gil · María Teresa López · María Asunción Lubiano · Manuel Montenegro · Gloria Naval · Antonia Salas

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Abstract Since Bertoluzza *et al.*'s metric between fuzzy numbers has been introduced, several studies involving it have been developed. Some of these studies concern equivalent expressions for the metric which are useful for either theoretical, practical or simulation purposes. Other studies refer to the potentiality of Bertoluzza *et al.*'s metric to establish statistical methods for the analysis of fuzzy data. This paper shortly reviews such studies and examine part of the scientific impact of the metric.

Keywords Bertoluzza *et al.*'s metric · fuzzy numbers · random fuzzy sets

1 Introduction

When analyzing fuzzy-valued data from a statistical perspective the use of suitable metrics between fuzzy data plays a crucial role.

On one hand, some of the main drawbacks associated with the difference operation can be often overcome by using distances. Thus, as for the usual fuzzy arithmetic there is no difference operation always well-defined and preserving the main properties of the real/vectorial-valued case, this operation can be replaced by a distance between fuzzy data when the 'sign' of the deviation is not relevant. On the other hand, distances are also essential in formalizing errors in estimating, statistical convergences in stating limit results, and so on.

* This paper is a modest tribute to our admired and beloved friend Professor Carlo Bertoluzza from the University of Pavia

Departamento de Estadística e I.O. y D.M., Universidad de Oviedo, 33007 Spain
E-mail: magil@uniovi.es

On quantifying the distance between fuzzy data two relevant features ought to be integrated, namely, the ease-to-handle and the intuitiveness of the interpretation. In this regard, Bertoluzza *et al.* [1] have introduced a generalized metric on the space of fuzzy numbers which is friendly to use and can be intuitively supported.

This paper aims to review Bertoluzza *et al.*'s generalized metric between fuzzy numbers, some equivalent expressions, as well as some topological properties. The choice of particular parameters/functions characterizing the metric is discussed. A concise review on some of the statistical methods for fuzzy data developed in this century and based on Bertoluzza *et al.*'s metric and the notion of random fuzzy sets [29] (originally coined as fuzzy random variables) is also given. The paper ends with some statistics on the scientific impact associated with Bertoluzza *et al.*'s metric [1].

2 Original definition, interpretation and metric properties

In the course of some studies on fuzzy regression analysis, Bertoluzza, Corral and Salas (Bertoluzza *et al.*) introduced in [1] a distance between fuzzy numbers extending the Euclidean one between real numbers.

By a *fuzzy number* (sometimes referred to as Zadeh's fuzzy number -see, for instance, Herencia and Lamata [18,19]-) we mean (see Goetschel and Voxman [11]) a fuzzy subset of the space of real numbers $\mathbb{R}$, that is, a mapping $\widetilde{U} : \mathbb{R} \to [0,1]$, which is convex, normal and upper semi-continuous with compact support.

Equivalently, a fuzzy number is a mapping $\widetilde{U} : \mathbb{R} \to [0,1]$ such that for each $\alpha \in [0,1]$, the α-level set (given by $\widetilde{U}_\alpha = \{x \in R : \widetilde{U}(x) \geq \alpha\}$ if $\alpha > 0$, $= \mathrm{cl}\{x \in R : \widetilde{U}(x) > 0\}$ if $\alpha = 0$) is a nonempty compact interval. $\widetilde{U}(x)$ is usually interpreted as the 'degree of compatibility of the real number x with the property associated with $\widetilde{U}$,' or 'degree of truth of the assertion "x is $\widetilde{U}$".'

Alternatively, Goetschel and Voxman [12] have proven that a fuzzy number is a mapping $\widetilde{U} : \mathbb{R} \to [0,1]$ such that

- $\inf \widetilde{U}_{(\cdot)} : [0,1] \to \mathbb{R}$ is a bounded non-decreasing function,
- $\sup \widetilde{U}_{(\cdot)} : [0,1] \to \mathbb{R}$ is a bounded non-increasing function,
- $\inf \widetilde{U}_1 \leq \sup \widetilde{U}_1$,
- $\inf \widetilde{U}_{(\cdot)}$ and $\sup \widetilde{U}_{(\cdot)}$ are left-continuous on $(0,1]$ and right-continuous at 0.

Let $\mathcal{F}_c(\mathbb{R})$ denote the space of fuzzy numbers. Bertoluzza *et al.* have suggested to compute the distance between two elements in $\mathcal{F}_c(\mathbb{R})$ "... as a suitable weighted mean of the distances between the α-levels of the fuzzy numbers." Consequently, "... the main difficulty is concerned with the definition of the distance between intervals,... so our first task consists on defining a measure of the distance between two intervals."

Bertoluzza *et al.* have pointed out some concerns related to the use of well-known distances on the space $\mathcal{K}_c(\mathbb{R})$ of the nonempty compact intervals, like Hausdorff L_∞-metric, which for $A, B \in \mathcal{K}_c(\mathbb{R})$ is given by

$$d_H(A, B) = \max\{|\inf A - \inf B|, |\sup A - \sup B|\},$$

or the L_p-metrics (see, for instance, Vitale [33]), which for $A, B \in \mathcal{K}_c(\mathbb{R})$ and $1 \leq p < \infty$ are given by

$$\delta_p(A, B) = \left(\frac{1}{2}\left|\inf A - \inf B\right|^p + \frac{1}{2}\left|\sup A - \sup B\right|^p\right)^{1/p}.$$

In this way, the fact that

$$d_H([0, 5], [6, 7]) = d_H([0, 5], [6, 10])$$

or

$$\delta_p([-2, 2], [-1, 1]) = \delta_p([-2, 1], [-1, 2]),$$

although in both cases the second intervals intuitively appear to be more distant, prevent from using these metrics in the statistical setting (and especially in the context of quantifying errors in estimation).

To overcome these drawbacks, when defining a new L_2-metric in $\mathcal{K}_c(\mathbb{R})$ Bertoluzza *et al.* suggest to involve not only the distances between the extreme values of the intervals, $|\inf A - \inf B|$ and $|\sup A - \sup B|$, but also those between other values of the intervals.

More concretely, to quantify the distance between intervals A and B

- A bijection between them is first established by associating for any arbitrary $t \in [0, 1]$: $A^{[t]} \leftrightarrow B^{[t]}$ (where $A^{[t]} = t \cdot \sup A + (1 - t) \cdot \inf A$);
- The root mean square Euclidean distance between the points associated through the bijection (see Figure 1), that is, $\left|A^{[t]} - B^{[t]}\right|^2$, is later computed.

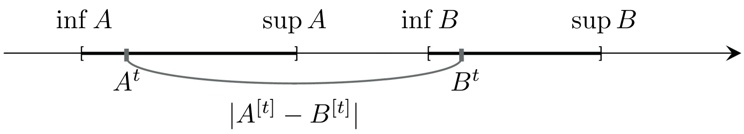

Fig. 1 The d_W-distance is given by a root mean square distance, the distance being the one which is graphically displayed, $|A^{[t]} - B^{[t]}|$

The suggested L_2-distance in $\mathcal{K}_c(\mathbb{R})$ is stated as follows:

Definition 1 *Let W be a normalized weighting measure on the measurable space $([0, 1], \mathcal{B}_{[0,1]})$ which is formalized as a probability measure associated with a non-degenerate distribution. The proposed distance is given for $A, B \in \mathcal{K}_c(\mathbb{R})$ by*

$$d_W(A, B) = \sqrt{\int_{[0,1]} \left|A^{[t]} - B^{[t]}\right|^2 dW(t)}.$$

Although the weighting measure W is formally associated with a probability measure, it has no stochastic but weighting meaning and mission. In particular, if W is associated with the uniform distribution on $\{0, 1\}$, then d_W reduces to δ_2. On the other hand, if W is associated with the uniform distribution on $[0, 1]$, which will be denoted along the paper by ℓ, then

$$d_\ell([0, 5], [6, 7]) = 4.1663 < 5.5076 = d_\ell([0, 5], [6, 10]),$$

$$d_\ell([-2, 2], [-1, 1]) = 0.5774 < 1 = d_\ell([-2, 1], [-1, 2]).$$

On extending this metric from $\mathcal{K}_c(\mathbb{R})$ to $\mathcal{F}_c(\mathbb{R})$, to quantify the distance between fuzzy numbers $\widetilde{U}$ and $\widetilde{V}$

– a double bijection between $\widetilde{U}$ and $\widetilde{V}$ is first established by associating
  - for any arbitrary $\alpha \in [0, 1]$: $\widetilde{U}_\alpha \leftrightarrow \widetilde{V}_\alpha$, and
  - for any arbitrary $t \in [0, 1]$: $\widetilde{U}_\alpha^{[t]} \leftrightarrow \widetilde{V}_\alpha^{[t]}$;
– the root mean square Euclidean distance between the points associated through the double bijection (see Figure 2), that is, $\left|\widetilde{U}_\alpha^{[t]} - \widetilde{V}_\alpha^{[t]}\right|^2$, is later computed.

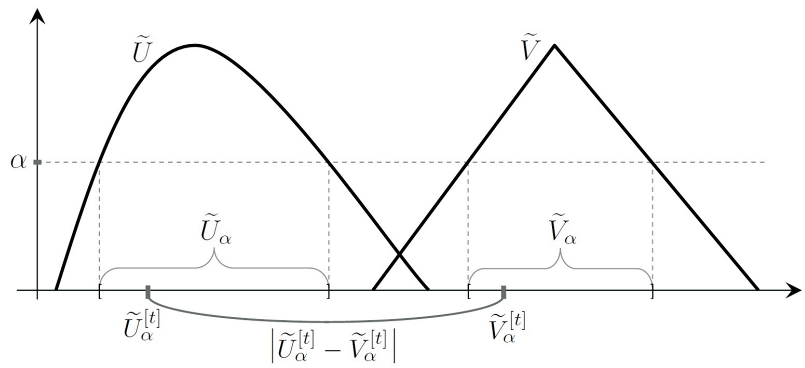

Fig. 2 The (W, φ)-distance is given by a root mean square distance, the distance being the one which is graphically displayed, $|\widetilde{U}_\alpha^{[t]} - \widetilde{V}_\alpha^{[t]}|$

The suggested L_2-distance in $\mathcal{F}_c(\mathbb{R})$ is stated as follows:

Definition 2 *Let W be a normalized weighting measure on the measurable space $([0, 1], \mathcal{B}_{[0,1]})$ which is formalized as a probability measure associated with a non-degenerate distribution, and φ be a normalized weighting measure on $([0, 1], \mathcal{B}_{[0,1]})$ which is formalized as a probability measure associated with an absolutely continuous distribution function being strictly increasing on $[0, 1]$. If $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$, then the (W, φ)-distance between two fuzzy numbers is given by the value*

$$D_W^\varphi(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left[d_W(\widetilde{U}_\alpha, \widetilde{V}_\alpha)\right]^2 \, d\varphi(\alpha)}$$

$$= \sqrt{\int_{[0,1]} \left[\int_{[0,1]} \left| \widetilde{U}_{\alpha}^{[t]} - \widetilde{V}_{\alpha}^{[t]} \right|^2 dW(t) \right] d\varphi(\alpha)}.$$

Although the weighting measure φ is formally associated with a probability measure, as it happens for W its meaning and mission are simply weighting but not stochastic in nature. Actually, φ weights the influence or importance of each level (i.e., degree of 'vagueness', 'fuzziness',...). Thus,

- if $\varphi \equiv \ell$, D_W^{φ} will be mainly sensitive to 'location' changes;
- if, for instance, $\varphi = \text{Beta}(1, p)$ with $p >> 1$ the lower the degree of compatibility, the higher the weight, whence D_W^{φ} will be very sensitive to changes at the lowest levels of compatibility;
- if, for instance, $\varphi = \text{Beta}(p, 1)$ with $p >> 1$ the higher the degree of compatibility, the higher the weight, whence D_W^{φ} will be very sensitive to changes at the highest levels of compatibility. Bertoluzza *et al.* have recommended that levels with high degree of compatibility should count more in the distance than those with low degree.

D_W^{φ} defines a metric on the space $\mathcal{F}_c(\mathbb{R})$. Thus,

Proposition 1 *D_W^{φ} is a metric on $\mathcal{F}_c(\mathbb{R})$.*

Proof Indeed,

- Non-negativity: trivial to prove.
- Identity of indiscernibles: As W is not associated with a degenerate distribution, $D_W^{\varphi}(\widetilde{U}, \widetilde{V}) = 0$ if, and only if,

$$\int_{[0,1]} \left| \widetilde{U}_{\alpha}^{[t]} - \widetilde{V}_{\alpha}^{[t]} \right|^2 dW(t) = 0 \quad a.s.\, [\varphi]$$

and, as φ is associated with an absolutely continuous distribution and $\left| \widetilde{U}_{\alpha}^{[t]} - \widetilde{V}_{\alpha}^{[t]} \right|^2$ is left-continuous at $\alpha \in (0, 1]$ and right-continuous at $\alpha = 0$, then $\int_{[0,1]} \left| \widetilde{U}_{\alpha}^{[t]} - \widetilde{V}_{\alpha}^{[t]} \right|^2 dW(t) = 0$ for all $\alpha \in [0, 1]$.
For any $\alpha \in [0, 1]$, since W is associated with a non-degenerate distribution, $\int_{[0,1]} \left| \widetilde{U}_{\alpha}^{[t]} - \widetilde{V}_{\alpha}^{[t]} \right|^2 dW(t) = 0$ implies that there exist two values $t_1(\alpha), t_2(\alpha) \in [0, 1]$, $t_1(\alpha) < t_2(\alpha)$, such that

$$t_1(\alpha)(\inf \widetilde{U}_{\alpha} - \inf \widetilde{V}_{\alpha}) + (1 - t_1(\alpha))(\sup \widetilde{U}_{\alpha} - \sup \widetilde{V}_{\alpha}) = 0,$$

$$t_2(\alpha)(\inf \widetilde{U}_{\alpha} - \inf \widetilde{V}_{\alpha}) + (1 - t_2(\alpha))(\sup \widetilde{U}_{\alpha} - \sup \widetilde{V}_{\alpha}) = 0,$$

and hence

$$\left[t_2(\alpha) - t_1(\alpha) \right] \cdot \left[(\inf \widetilde{U}_{\alpha} - \inf \widetilde{V}_{\alpha}) - (\sup \widetilde{U}_{\alpha} - \sup \widetilde{V}_{\alpha}) \right] = 0.$$

In case either $t_1(\alpha)$ or $t_2(\alpha)$ belong to $(0, 1)$, the unique possibility for the three preceding equalities to hold is that

$$\inf \widetilde{U}_{\alpha} = \inf \widetilde{V}_{\alpha}, \;\; \sup \widetilde{U}_{\alpha} = \sup \widetilde{V}_{\alpha}.$$

In case $t_1(\alpha) = 0$ and $t_2(\alpha) = 1$, then also the unique possibility for the two first preceding equalities to hold is that

$$\inf \widetilde{U}_\alpha = \inf \widetilde{V}_\alpha, \ \ \sup \widetilde{U}_\alpha = \sup \widetilde{V}_\alpha.$$

Consequently, $\widetilde{U} = \widetilde{V}$.

- Symmetry: trivial to prove.
- Triangular inequality: quite trivial to prove because of

$$\left|\widetilde{U}_\alpha^{[t]} - \widetilde{V}_\alpha^{[t]}\right|^2 \leq \left|\widetilde{U}_\alpha^{[t]} - \widetilde{T}_\alpha^{[t]}\right|^2 + \left|\widetilde{T}_\alpha^{[t]} - \widetilde{V}_\alpha^{[t]}\right|^2$$

for all $\alpha \in [0, 1]$ and $\widetilde{U}, \widetilde{V}, \widetilde{T} \in \mathcal{F}_c(\mathbb{R})$. □

Remark 1 It should be pointed out that for the developments in [1] authors restrict W to be a mixture of a discrete-finite distribution and a continuous one, but in fact there is no need for such a constraint in the general setting. Similarly, the absolute continuity of φ could be weakened by simply demanding a condition guaranteeing the identity of indiscernibles for D_W^φ, but the assumed condition seems to be ease-to-use and rather natural in practice.

3 Definitional and topological equivalences

As it has been detailed in previous studies (see, for instance, Blanco-Fernández *et al.* [3]), the metric D_W^φ can be alternatively expressed in some different ways.

The expression as it was introduced by Bertoluzza *et al.* is definitely the easiest version to interpret as it involves the choice of the weighting measure W. Nevertheless, for computations, simulations, theoretical developments and the extension to higher dimensionality spaces, some equivalences become more appropriate. These 'definitional' equivalences has been also described in Blanco-Fernández *et al.* [3].

3.1 *Equivalent definition based on weighting extremes and a relevant location point of the α-levels*

As eventually happens in Maths, the generalized metric in Definition 2 can be equivalently characterized by means of one of its particularizations.

Thus, D_W^φ can be fully characterized (see Bertoluzza *et al.* [1], Lubiano *et al.* [23]) by particularizing the general weighting measure W to a discrete one weighting (for each level) at three points: the two extremes and an intermediate one (often the mid-point).

Thus, given W and φ, if one denotes $t_W = \int_{[0,1]} t \, dW(t)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ with

$$\lambda_1 = \frac{\displaystyle\int_{[0,1]} (t - t_W)^2 \, dW(t)}{1 - t_W}, \ \lambda_2 = \frac{\displaystyle\int_{[0,1]} t(1-t) \, dW(t)}{t_W(1 - t_W)}, \ \lambda_3 = \frac{\displaystyle\int_{[0,1]} (t - t_W)^2 \, dW(t)}{t_W},$$

then,

$$D_W^{\varphi}(\widetilde{U},\widetilde{V}) = D_{\boldsymbol{\lambda}}^{\varphi}(\widetilde{U},\widetilde{V})$$

$$= \sqrt{\int_{(0,1]} \left(\lambda_1 \left[\widetilde{U}_\alpha^{[1]} - \widetilde{V}_\alpha^{[1]}\right]^2 + \lambda_2 \left[\widetilde{U}_\alpha^{[t_W]} - \widetilde{V}_\alpha^{[t_W]}\right]^2 + \lambda_3 \left[\widetilde{U}_\alpha^{[0]} - \widetilde{V}_\alpha^{[0]}\right]^2\right) d\varphi(\alpha)}.$$

It can be easily verified that $\lambda_1 > 0, \lambda_2 \geq 0, \lambda_3 > 0$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Moreover, if $t_W = 0.5$ (like it happens, for instance, if W is associated with a symmetric distribution w.r.t. $t = 0.5$, which is often a reasonable selection), then $\widetilde{U}_\alpha^{[t_W]} = \operatorname{mid}\widetilde{U}_\alpha = \text{centre of } \widetilde{U}_\alpha = (\inf \widetilde{U}_\alpha + \sup \widetilde{U}_\alpha)/2$.

Although choosing W is more intuitive than choosing the weighting vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, the last one would be easy-to-handle in many other developments. Some possible selections for the weighting vector with the correspondent measure W are gathered in Table 1.

W	$\boldsymbol{\lambda}$
$\text{Beta}(p,q)$	$\left(\frac{p}{(p+q)(p+q+1)}, \frac{p+q}{p+q+1}, \frac{q}{(p+q)(p+q+1)}\right)$
$\text{Uniform}\left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}$	$\left(\frac{k+2}{6k}, \frac{2k-2}{3k}, \frac{k+2}{6k}\right)$
$\dfrac{\text{Binom}(k,p)}{k}$	$\left(\frac{p}{k}, \frac{k-1}{k}, \frac{1-p}{k}\right)$

Table 1 Some possible choices for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ which are based on choices for W

3.2 *Equivalent definition based on weighting the centers and radii of the α-levels*

It is well-known that Hausdorff's metric can be equivalently expressed in the interval-valued case as

$$d_H(A,B) = |\operatorname{mid} A - \operatorname{mid} B| + |\operatorname{spr} A - \operatorname{spr} B|,$$

where $\operatorname{mid} A = (\inf A + \sup A)/2$ = centre (mid-point) of A, $\operatorname{spr} A = (\sup A - \inf A)/2$ = radius (spread) of A.

In a similar way, D_W^{φ} can also be expressed (see, for instance, Gil *et al.* [8,9], Trutschnig *et al.* [32]) by replacing the general weighting measure W by a non-stochastic weighting of the squared distances between the intermediate points associated with t_W and the squared distances between the radii. Thus, given W and φ, if one denotes $\theta = 4\int_{[0,1]}(t - t_W)^2\, dW(t) = 4\,\lambda_1(1 - t_W) \in (0,1]$, then

$$D_W^{\varphi}(\widetilde{U},\widetilde{V}) = D_\theta^{\varphi}(\widetilde{U},\widetilde{V})$$

$$= \sqrt{\int_{[0,1]} \left(\left[\widetilde{U}_\alpha^{[t_W]} - \widetilde{V}_\alpha^{[t_W]}\right]^2 + \theta\left[\operatorname{spr}\widetilde{U}_\alpha - \operatorname{spr}\widetilde{V}_\alpha\right]^2\right) d\varphi(\alpha)}.$$

If $t_W = 0.5$ (in particular, if W is associated with a symmetric distribution w.r.t. 0.5), then

$$D_\theta^\varphi(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left(\left[\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha\right]^2 + \theta \left[\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha\right]^2 \right) d\varphi(\alpha)}.$$

Consequently, the choices of W and θ allow us to weight for each α the effect of the deviation in 'shape/imprecision' in contrast to the effect of the deviation in 'location/position'. From a theoretical perspective we could extend the parameter θ to range on $(0, \infty)$, but in practice it seems more reasonable to constrain θ to $(0, 1]$ so that the deviation in shape/imprecision is weighted up to the deviation in location.

Although choosing W is more intuitive than choosing the weighting parameter θ, the last one would be easy-to-handle in many other developments. Some possible selections for the weighting parameter with the correspondent measure W are gathered in Table 2.

W	θ
$\text{Beta}(p, q)$	$\frac{4pq}{(p+q)^2(p+q+1)}$
$\text{Uniform}\left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}$	$\frac{k+2}{3k}$
$\frac{\text{Binom}(k, p)}{k}$	$\frac{4p(1-p)}{k}$

Table 2 Some possible choices for θ which are based on choices for W

3.3 *Equivalent definition based on the support functions of the fuzzy numbers*

Fuzzy numbers (in general, convex fuzzy sets) can also be characterized by means of the so-called support function (see Puri and Ralescu [28]). If $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$, the support function of $\widetilde{U}$ is the real valued function $s_{\widetilde{U}}$ on $\{-1, 1\} \times [0, 1]$ such that $s_{\widetilde{U}}(-1, \alpha) = -\inf \widetilde{U}_\alpha$ and $s_{\widetilde{U}}(1, \alpha) = \sup \widetilde{U}_\alpha$.

By using this function, D_W^φ can be expressed (see Näther [27], Körner and Näther [21]) by replacing the general weighting measure W by a definite positive and symmetric kernel K defined on $\{-1, 1\}^2 \times [0, 1]^2$ such that

$$dK(u, v, \alpha, \beta) = \begin{cases} K^0(u, v)\, d\varphi(\alpha) & \text{if } \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

with

$$K^0(1, 1) = \int_{[0,1]} t^2\, dW(t) = \lambda_1(1 - t_W) + t_W^2,$$

$$K^0(-1,-1) = \int_{[0,1]} (1-t)^2 \, dW(t) = \lambda_1(1-t_W) + (1-t_W)^2,$$

$$K^0(1,-1) = K^0(-1,1) = \int_{[0,1]} t(1-t) \, dW(t) = (t_W - \lambda_1)(1-t_W).$$

Thus, given W and φ, by considering the inner product $\langle \cdot, \cdot \rangle_K$ associated with the L^2-distance on the space of the Lebesgue integrable functions on $\{-1,1\} \times [0,1]$ w.r.t. the above definite positive and symmetric kernel K, we have that

$$D_W^{\varphi}(\widetilde{U}, \widetilde{V}) = D_K^{\varphi}(\widetilde{U}, \widetilde{V}) = \sqrt{\langle s_{\widetilde{U}} - s_{\widetilde{V}}, s_{\widetilde{U}} - s_{\widetilde{V}} \rangle_K}$$

$$= \sqrt{\int_{(\{-1,1\}\times[0,1])^2} (s_{\widetilde{U}}(u,\alpha) - s_{\widetilde{V}}(u,\alpha))(s_{\widetilde{U}}(v,\beta) - s_{\widetilde{V}}(v,\beta)) \, dK(u,v,\alpha,\beta)}.$$

Although choosing W is more intuitive than choosing the definite positive and symmetric kernel K, the latter would be convenient for certain developments, as we will see in the next section. Some possible selections for the kernel with the correspondent measure W are gathered in Table 3.

W	$\begin{pmatrix} K^0(1,1) & K^0(1,-1) \\ K^0(-1,1) & K^0(-1,-1) \end{pmatrix}$
$\mathrm{Beta}(p,q)$	$\begin{pmatrix} \frac{p(1+p)}{(p+q)(p+q+1)} & \frac{pq}{(p+q)(p+q+1)} \\ \frac{pq}{(p+q)(p+q+1)} & \frac{q(1+q)}{(p+q)(p+q+1)} \end{pmatrix}$
$\mathrm{Uniform}\left\{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\right\}$	$\begin{pmatrix} \frac{2k+1}{6k} & \frac{k-1}{6k} \\ \frac{k-1}{6k} & \frac{2k+1}{6k} \end{pmatrix}$
$\dfrac{\mathrm{Binom}(k,p)}{k}$	$\begin{pmatrix} \frac{p[(1-p)+kp]}{k} & \frac{p(1-p)(k-1)}{k} \\ \frac{p(1-p)(k-1)}{k} & \frac{(1-p)[p+k(1-p)]}{k} \end{pmatrix}$

Table 3 Some possible choices for the definite positive and symmetric kernel which are based on choices for W

As a summary implication of the equivalences which have been just stated, Table 4 jointly collects some particular choices of the weighting $\boldsymbol{\lambda}$, θ and $\begin{pmatrix} K^0(1,1) & K^0(1,-1) \\ K^0(-1,1) & K^0(-1,-1) \end{pmatrix}$ for certain symmetric selections of W (the symmetric being usually the most natural ones).

On the other hand, Bertoluzza *et al.*'s metric is topologically equivalent to well-known separable metrics, which leads to valuable features for D_W^{φ}, as can be seen in the following subsection.

W	$\boldsymbol{\lambda}$	θ	$\begin{pmatrix} K^0(1,1) & K^0(1,-1) \\ K^0(-1,1) & K^0(-1,-1) \end{pmatrix}$
$\mathrm{Beta}(1,1) = \ell$	$(1/6, 2/3, 1/6)$	$1/3$	$\begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix}$
$\mathrm{Beta}(2,2)$	$(1/10, 4/5, 1/10)$	$1/5$	$\begin{pmatrix} 3/10 & 1/5 \\ 1/5 & 3/10 \end{pmatrix}$
$\mathrm{Beta}(1/4,1/4)$	$(1/3, 1/3, 1/3)$	$2/3$	$\begin{pmatrix} 5/12 & 1/12 \\ 1/12 & 5/12 \end{pmatrix}$
$\mathrm{Beta}(1/8,1/8)$	$(2/5, 1/5, 2/5)$	$4/5$	$\begin{pmatrix} 9/20 & 1/20 \\ 1/20 & 9/20 \end{pmatrix}$
$\mathrm{Uniform}\{0, 1/2, 1\}$	$(1/3, 1/3, 1/3)$	$2/3$	$\begin{pmatrix} 5/12 & 1/12 \\ 1/12 & 5/12 \end{pmatrix}$
$\mathrm{Binom}(4, 1/2)/4$	$(1/8, 3/4, 1/8)$	$1/4$	$\begin{pmatrix} 5/16 & 3/16 \\ 3/16 & 5/16 \end{pmatrix}$

Table 4 Some particular choices for the weighting vector, parameter and definite positive and symmetric kernel which are based on choices for W

3.4 *Topological properties*

Bertoluzza *et al.*'s metric is topologically equivalent to the L^2-metric ρ_2 between fuzzy numbers based on δ_2 (Diamond and Kloeden [7]), which is given by

$$\rho_2(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left[\delta_2(\widetilde{U}_\alpha, \widetilde{V}_\alpha)\right]^2 \, d\alpha}$$

$$= \sqrt{\int_{[0,1]} \left[\frac{1}{2} \cdot \left|\inf \widetilde{U}_\alpha - \inf \widetilde{V}_\alpha\right|^2 + \frac{1}{2} \cdot \left|\sup \widetilde{U}_\alpha - \sup \widetilde{V}_\alpha\right|^2\right] d\alpha},$$

and can be easily extended to

$$\rho_2^\varphi(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left[\frac{1}{2} \cdot \left|\inf \widetilde{U}_\alpha - \inf \widetilde{V}_\alpha\right|^2 + \frac{1}{2} \cdot \left|\sup \widetilde{U}_\alpha - \sup \widetilde{V}_\alpha\right|^2\right] d\varphi(\alpha)}.$$

Whenever $\theta \in (0,1]$, the last metric is equivalent to D_θ^φ. Thus,

Proposition 2 *Let φ be a normalized weighting measure on the measurable space $([0,1], \mathcal{B}_{[0,1]})$ which is formalized as a probability measure associated with an absolutely continuous distribution function being strictly increasing on $[0,1]$, and let $\theta \in (0,1]$. The metric D_θ^φ is topologically equivalent to the metric ρ_2^φ on $\mathcal{F}_c(\mathbb{R})$. More precisely,*

$$\sqrt{\theta} \cdot \rho_2^\varphi(\widetilde{U}, \widetilde{V}) \leq D_\theta^\varphi(\widetilde{U}, \widetilde{V}) \leq \rho_2^\varphi(\widetilde{U}, \widetilde{V})$$

for all $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$.

Proof Indeed, for each $\alpha \in [0,1]$ and $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ it is obvious that

$$\theta \cdot [\delta_2(\widetilde{U}_\alpha, \widetilde{V}_\alpha))]^2 = \theta \cdot \left|\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha\right|^2 + \theta \cdot \left|\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha\right|^2$$

$$\leq \left|\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha\right|^2 + \theta \cdot \left|\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha\right|^2$$

$$\leq \left|\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha\right|^2 + \left|\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha\right|^2 = [\delta_2(\widetilde{U}_\alpha, \widetilde{V}_\alpha))]^2.$$

Since

$$\delta_2(\widetilde{U}_\alpha, \widetilde{V}_\alpha) = d_{\text{Uniform}\{0,1\}}(\widetilde{U}_\alpha, \widetilde{V}_\alpha)$$

$$= \sqrt{\left|\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha\right|^2 + \left|\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha\right|^2},$$

then,

$$\sqrt{\theta} \cdot \rho_2^\varphi(\widetilde{U}, \widetilde{V}) \leq D_\theta^\varphi(\widetilde{U}, \widetilde{V}) \leq \rho_2^\varphi(\widetilde{U}, \widetilde{V}).$$

Therefore, D_θ^φ and ρ_2^φ are topologically equivalent. □

Given that ρ_2^φ is topologically equivalent to d_2^φ, which extends the L^2-metric d_2 in Diamond and Kloeden as follows:

$$d_2^\varphi(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left[d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha)\right]^2 \, d\varphi(\alpha)},$$

D_θ^φ, ρ_2^φ and d_2^φ share all the topological advantages of the last one, separability among them. Thus, by following arguments similar to those in Diamond and Kloeden [7], the separability of the metric space $(\mathcal{F}_c(\mathbb{R}), d_2^\varphi)$ can be proved and, hence,

Proposition 3 $(\mathcal{F}_c(\mathbb{R}), D_W^\varphi)$ *is a separable metric space.*

Although D_W^φ and $D_{\boldsymbol{\lambda}}^\varphi$ have been the first versions of Bertoluzza *et al.*'s metric, D_K^φ and D_θ^φ have been preferred for most statistical developments. Some of the arguments supporting such a preference (see [3]) are the following:

- the mid/spread representation of fuzzy numbers provides some valuable results, especially in connection with regression studies;
- D_K^φ and D_θ^φ can be extended to fuzzy sets of higher dimension Euclidean spaces (see Näther [27], Körner and Näther [21] for the extension of D_K^φ, and Trutschnig *et al.* [32] for the extension of D_θ^φ) on the basis of the support function of fuzzy sets (Puri and Ralescu [28]), which is an alternative characterization of fuzzy sets with compact convex levels through their boundaries;
- the covariance of two random mechanisms producing fuzzy data can be formalized prompted by the ideas for generalized space-valued random elements.

4 Some applications to the statistical analysis of fuzzy data

The arithmetic of fuzzy numbers is a basic tool for statistically analyzing fuzzy data. More concretely, the the sum of fuzzy numbers and the product of a real by a fuzzy number are the key operations in this setting.

The usual arithmetic to be considered on $\mathcal{F}_c(\mathbb{R})$ is that based on Zadeh's extension principle [34], which level-wise inherits the usual and natural interval arithmetic, that is, for $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ and $\gamma \in \mathbb{R}$ the *sum* $\widetilde{U} + \widetilde{V}$ is the fuzzy number such that for each $\alpha \in [0,1]$

$$(\widetilde{U} + \widetilde{V})_\alpha = \text{Minkowski sum of } \widetilde{U}_\alpha \text{ and } \widetilde{V}_\alpha$$

$$= [\inf \widetilde{U}_\alpha + \inf \widetilde{V}_\alpha, \sup \widetilde{U}_\alpha + \sup \widetilde{V}_\alpha],$$

and the *product by the scalar* $\gamma \cdot \widetilde{U}$ is the fuzzy number such that for each $\alpha \in [0,1]$

$$(\gamma \cdot \widetilde{U})_\alpha = \gamma \cdot \widetilde{U}_\alpha = \begin{cases} \left[\gamma \cdot \inf \widetilde{U}_\alpha, \gamma \cdot \sup \widetilde{U}_\alpha\right] & \text{if } \gamma \geq 0 \\ \left[\gamma \cdot \sup \widetilde{U}_\alpha, \gamma \cdot \inf \widetilde{U}_\alpha\right] & \text{otherwise.} \end{cases}$$

When the metric D_W^φ is combined with the usual fuzzy arithmetic it can be concluded that it is translational invariant, i.e.,

$$D_W^\varphi(\widetilde{U} + \widetilde{T}, \widetilde{V} + \widetilde{T}) = D_W^\varphi(\widetilde{U}, \widetilde{V}),$$

and in case W is associated with a symmetric distribution on $[0,1]$ (more generally, in case and only in case $t_W = 0.5$) it is also rotational invariant, i.e.,

$$D_W^\varphi((-1) \cdot \widetilde{U}, (-1) \cdot \widetilde{V}) = D_W^\varphi(\widetilde{U}, \widetilde{V}).$$

Random fuzzy sets is another basic tool for the analysis of fuzzy data, especially to support appropriately the methods of analysis within a probabilistic framework. This concept was originally coined by Puri and Ralescu [29] as fuzzy random variables. Random fuzzy sets mean a mathematical model for the random mechanism generating fuzzy data.

In the one-dimensional case, $\mathcal{F}_c(\mathbb{R})$, a *random fuzzy set* (or random fuzzy number) is formalized as follows:

Definition 3 *Given a probability space $(\Omega, \mathcal{A}, P)$, a random fuzzy number associated with it is a mapping $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ such that the α-level mappings $\mathcal{X}_\alpha : \Omega \to \mathcal{K}_c(\mathbb{R})$, with $\mathcal{X}_\alpha(\omega) = (\mathcal{X}(\omega))_\alpha$ for every $\alpha \in [0,1]$, is a random interval.*

A random fuzzy number can be proven to be Borel-measurable w.r.t. the Borel σ-field generated by the topology induced by D_W^φ (see, for instance, González-Rodríguez *et al.* [15]). Consequently, one can trivially refer to the *induced distribution of a random fuzzy number*, the *independence of two fuzzy numbers*, and so on.

The *Aumann-type mean of* $\mathcal{X}$ is one of the most valuable measures to summarize the information in the distribution of a random fuzzy number. If it exists, it is defined as the unique $\widetilde{E}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R})$ such that for each $\alpha \in [0,1]$

$$\left(\widetilde{E}(\mathcal{X})\right)_\alpha = [E(\inf \mathcal{X}_\alpha), E(\sup \mathcal{X}_\alpha)].$$

This notion is coherent with fuzzy arithmetic, so that if

$$\mathcal{X} = \mathbb{1}_{\widetilde{x}_1} \cdot \widetilde{x}_1 + \ldots + \mathbb{1}_{\widetilde{x}_r} \cdot \widetilde{x}_r,$$

where $\widetilde{x}_i \in \mathcal{F}_c(\mathbb{R})$ $(i = 1, \ldots, r)$ and $\mathbb{1}$ denoting the indicator function in Ω, then,

$$\widetilde{E}(\mathcal{X}) = P(\mathcal{X} = \widetilde{x}_1) \cdot \widetilde{x}_1 + \ldots + P(\mathcal{X} = \widetilde{x}_r) \cdot \widetilde{x}_r.$$

Moreover, $\widetilde{E}(\mathcal{X})$ is coherent with Fréchet's principle for D_W^φ, that is,

$$\widetilde{E}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathcal{F}_c(\mathbb{R})} E\left(\left[D_W^\varphi(\mathcal{X}, \widetilde{U})\right]^2\right).$$

$\widetilde{E}(\mathcal{X})$ is supported by different Strong Laws of Large Numbers (see, for instance, Colubi *et al.* [6]).

Other relevant summary measures of the distribution of a random fuzzy number are the Fréchet variance based on D_W^φ (see, for instance, Lubiano *et al.* [23], Blanco-Fernández *et al.* [2]), and the L^1-medians by Sinova *et al.* [30, 31]. The covariance of two random fuzzy numbers can be also introduced (see González-Rodríguez *et al.* [13], Blanco-Fernández *et al.* [2]) in connection with the simple linear regression analysis between random fuzzy sets, although in this case it does not involve D_W^φ but is based on the support function.

Estimating the population fuzzy-valued Aumann-type mean of $\mathcal{X}$ on the basis of a sample of independent observations from it is one of the statistical problems in which Bertoluzza *et al.*'s metric is involved. More concretely (see, for instance, Lubiano and Gil [22], González-Rodríguez *et al.* [17], Blanco-Fernández *et al.* [2]),

- in what concerns to the *'point' estimation* of $\widetilde{E}(\mathcal{X})$, the metric D_W^φ is used to quantify the estimation error;
- in what concerns to the *'confidence' estimation* of $\widetilde{E}(\mathcal{X})$, D_W^φ is used to construct a confidence ball of this fuzzy-valued parameter.

Another statistical problem involving Bertoluzza *et al.*'s metric is that of testing about the population fuzzy-valued Aumann-type mean of one or more random fuzzy numbers on the basis of a sample of independent observations from it or them. More concretely (see Körner [20], Montenegro *et al.* [25,26], González-Rodríguez *et al.* [16,15], Gil *et al.* [10], and Blanco-Fernández *et al.* [2]),

– in what concerns the *one-sample testing of the null hypothesis* $H_0 : \widetilde{E}(\mathcal{X}) = \widetilde{U}$ the metric D_W^{φ} is used to test the equivalent null hypothesis $H_0 : D_W^{\varphi}(\widetilde{E}(\mathcal{X}), \widetilde{U}) = 0$;
– in what concerns the *two-sample testing of the null hypothesis* $H_0 : \widetilde{E}(\mathcal{X}) = \widetilde{E}(\mathcal{Y})$ the metric D_W^{φ} is used to test the equivalent null hypothesis $H_0 : D_W^{\varphi}(\widetilde{E}(\mathcal{X}), \widetilde{E}(\mathcal{Y})) = 0$ for independent and dependent samples;
– in what concerns the *k-sample testing of the null hypothesis* $H_0 : \widetilde{E}(\mathcal{X}_1) = \ldots = \widetilde{E}(\mathcal{X}_m)$ the metric D_W^{φ} is used to test the equivalent null hypothesis $H_0 : \sum_{i=1}^{m} \left[D_W^{\varphi}\big(\widetilde{E}(\mathcal{X}_i), \frac{1}{m}(\mathcal{X}_1 + \ldots + \mathcal{X}_m)\big)\right]^2 = 0$ for independent and dependent samples.

A third statistical problem in which Bertoluzza *et al.*'s metric has been shown to be useful is that of the *linear regression analysis between two random fuzzy numbers* (see, for instance, González-Rodríguez *et al.* [13], Blanco-Fernández *et al.* [2]). D_W^{φ} has been employed to develop a least squares approach to solve the linear regression problem when the usual fuzzy arithmetic is considered.

A fourth problem using D_W^{φ} is that of *classifying fuzzy data* (see Colubi *et al.* [5], Blanco-Fernández *et al.* [2]). The metric has been considered to compute the distance between the fuzzy data to be classified and the set of training fuzzy data.

An R package (`http://cran.r-project.org/web/packages/SAFD/index.html`) has been designed, and it is being continuously updated, by Lubiano and Trutschnig (see, for instance, Lubiano and Trutschnig [24]). It provides several basic functions to carry out statistics with one-dimensional fuzzy data in accordance with the statistical methodology based on Bertoluzza *et al.*'s metric.

5 Analyzing the impact of Bertoluzza *et al.*'s distance

To end this paper an elementary statistical analysis is to be considered in connection with the impact of Bertoluzza *et al.*'s distance. For this purpose, we have examined three scientific databases, namely, the *Web of Science* (Thomson Reuters), *SCOPUS* (Elsevier) and *Google Scholar*.

It should be highlighted that *Mathware & Soft Computing*, the journal Bertoluzza *et al.*'s distance has been published in, has not yet entered the two first databases. However, the three databases include the number of citations the paper has received, this number varying among the databases because of the type of documents they cover.

Table 5 shows the number of citations per periods of three years in accordance with the three databases (notice that the number of citations in the last considered period is likely to increase since the period is not yet ended). It can easily be concluded from the table that this number is rather increasing, which means the concept is being widely used.

	1999–2001	2002–2004	2005–2007	2008–2010	2010–2013	**Total**
Web of Science	10	8	14	20	17	**69**
SCOPUS	10	6	15	23	18	**72**
Google Scholar	11	13	24	30	40	**118**

Table 5 Citations received by [1] Bertoluzza, C., Corral, N., Salas, A.: On a new class of distances between fuzzy numbers. Mathw. & Soft Comp. **2** 71-84 (1995), in accordance with *Web of Science*, *SCOPUS* and *Google Scholar*

The citations have been classified in different categories. The eight first (by citations number) according to the Web of Science classification) are shown in Figure 3. Most of them correspond to STATISTICS & PROBABILITY, branch which was the original motivation for introducing the distance. It has also been widely applied in COMPUTER SCIENCE and MATHEMATICS.

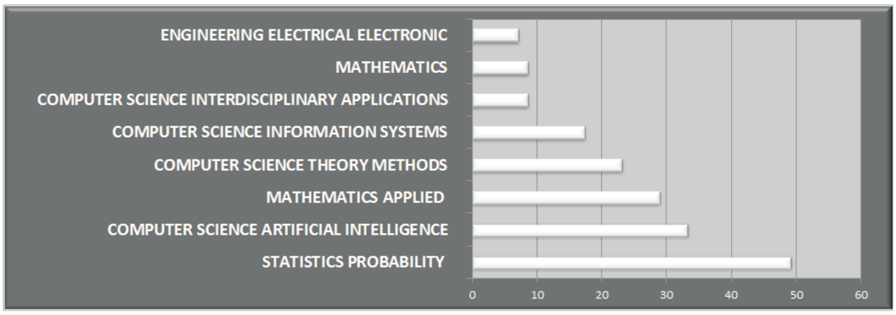

Fig. 3 Distribution (percentages) of the papers citing [1] Bertoluzza, C., Corral, N., Salas, A.: On a new class of distances between fuzzy numbers. Mathw. & Soft Comp. **2** 71-84 (1995) by Web of Science Categories (eight first categories) (Source: Web of Science)

Figure 4 shows the six first (by citations number) journals the citations have been published in. The paper has been also cited in chapters of multi-authors books published by Springer and included in the WoS.

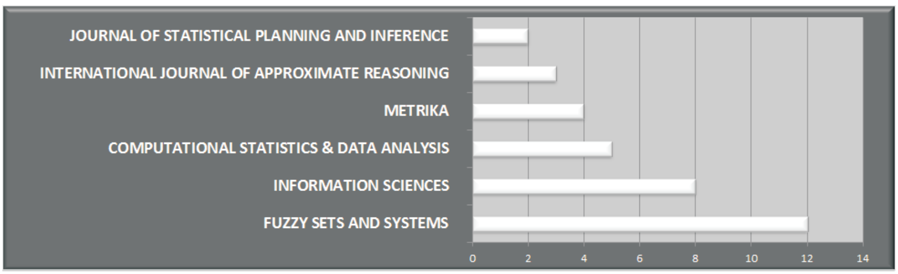

Fig. 4 Distribution of the journals papers citing [1] Bertoluzza, C., Corral, N., Salas, A.: On a new class of distances between fuzzy numbers. Mathw. & Soft Comp. **2** 71-84 (1995) by journals (six first ones) (Source: Web of Science)

Finally, Figure 5 shows the first (by citations number) countries the authors institutions belong to.

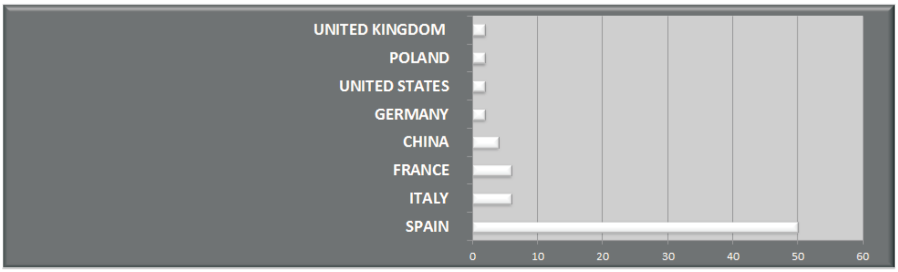

Fig. 5 Distribution of the papers citing [1] Bertoluzza, C., Corral, N., Salas, A.: On a new class of distances between fuzzy numbers. Mathw. & Soft Comp. **2** 71-84 (1995) by countries (eight first ones) (Source: Web of Science)

These figures prove an increasing interest on [1], so we foresee that, in a few years, the numbers in the last two figures will substantially increase.

6 Concluding remarks

In this paper we have presented a review on how Bertoluzza *et al.*'s metric has been applied aiming to analyze fuzzy data generated through a random process.

It should be mentioned that this metric between fuzzy numbers can also be considered in order to test about distributions of real-valued random variables. We can state a statistical distance between probability distributions of real-valued random variables on the basis of the so-called characterizing fuzzy representation of a random variable (see González-Rodríguez *et al.* [14], and also Blanco-Fernández *et al.* in this issue [4]). This distance is given by Bertoluzza *et al.*'s one between the Aumann-type means of the characterizing fuzzy representations of these distributions. It can be used for estimating the distribution of a random variable, for Goodness-of-Fit testing, and for testing the equality of two or more distributions. The corresponding estimation and testing procedures derive from the particularization of the estimators of/tests about means of random fuzzy numbers we have succinctly recalled in Section 4.

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The authors wish to mention that they are the ‘oldest’ representatives of the research group SMIRE of the University of Oviedo (Spain) in which all the members have, before or later, considered Bertoluzza *et al.*'s metric in their research. Therefore, this paper can be also intended to be a tribute from the ‘youngest’ part of the group, namely, Ángela Blanco-Fernández, Ana Colubi, Sara de la Rosa de Sáa, Marta García-Bárzana, Gil González-Rodríguez, Ana Belén Ramos-Guajardo and Beatriz Sinova.

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